EXERCISES

Part I

Consider the set of Maxwell's equations in vacuum:

$$(1) \qquad \frac{\partial \mathbf{E}}{\partial t} = c^2 \, \mathrm{curl}\mathbf{B}$$

$$(2) \qquad \mathrm{div}\mathbf{E} = 0$$

$$(3) \qquad \frac{\partial \mathbf{B}}{\partial t} = - \, \mathrm{curl}\mathbf{E}$$

$$(4) \qquad \mathrm{div}\mathbf{B} = 0$$

In the Cartesian reference frame (x, y, z), take the following solution candidates:

$$(5) \qquad \mathbf{E} = \Big(E_1(x,y), \; E_2(x,y), \; 0\Big) g(z - ct)$$

$$\mathbf{B} = \Big(B_1(x,y), \; B_2(x,y), \; 0\Big) g(z - ct)$$

representing two field distributions modulated in time by the function g. The fields are transverse and lay on parallel planes shifting in the direction of the z-axis at the speed of light c. The functions E_1, E_2, B_1 and B_2 are supposed to be smooth on the whole plane (x, y). They may be allowed to be zero outside a domain of finite measure, but this has to be done without discontinuities, i.e., the functions must smoothly go to zero at the border of the domain, so that they can be continuously prolonged to zero. The function g is also smooth.

1. Show that, for any g, the equations (1) and (3) imply:

$$(6) \qquad E_1 = cB_2 \qquad E_2 = -cB_1 \qquad B_{2,x} = B_{1,y} \qquad E_{2,x} = E_{1,y}$$

where the subscripts x and y denote partial derivative with respect to such variables.

2. Deduce that equations (2) and (4) are automatically satisfied. Moreover, one has that $\mathbf{E}$ and $\mathbf{B}$ are orthogonal with $\|\mathbf{E}\| = \|c\mathbf{B}\|$.

3. Deduce that the functions $E_1 - iE_2$ and $B_1 - iB_2$, where i is the imaginary unit, are holomorphic (entire) on the whole complex plane $x+iy$ (see, for instance [1]).

4. Use the Liouville's theorem: *if an entire holomorphic function is bounded, then it is a constant*, to entail that there are not bounded continuous electromagnetic fields of the form (5) having finite energy and solving the whole set of Maxwell's equations. The only exception is the trivial choice $\mathbf{E} = 0$ and $\mathbf{B} = 0$.

5. Try to generalize the setting in (5) by introducing the third components $E_3(x, y)$ and $B_3(x, y)$. By analyzing the third component of (1) and (2), show, using separation of variables arguments, that g must be a real exponential. Therefore, if g is zero in one point (the 'head' of the advancing front, for instance), then g must be zero everywhere.

6. Try to generalize the setting in (5) by using several functions g_i, $i = 1, 2, 3, 4$ as follows:

$$\mathbf{E} = \Big(E_1(x, y)g_1(z - ct),\ E_2(x, y)g_2(z - ct),\ 0\Big) \tag{7}$$

$$\mathbf{B} = \Big(B_1(x, y)g_3(z - ct),\ B_2(x, y)g_4(z - ct),\ 0\Big)$$

Prove that $g_1 = g_4$ and $g_2 = g_3$. Recover (6) and then proceed starting from point **1**.

Moral: The only possible solutions of Maxwell's equations in vacuum, having bounded electromagnetic fields laying on flat wave-fronts, are either identically zero or have infinite energy. Note that solutions may exist if we assume that **E** and **B** do not belong to the tangent plane of the advancing front. In this situation, however, the Poynting's vector $\mathbf{E}\times\mathbf{B}$, indicating the direction of the energy flow is not lined up with the direction of movement. These solutions, displaying a complicate behavior, are not obtainable by separating the time and space variables as done in (5). Moreover, they do not satisfy the rules of geometrical optics.

Part II

Transform the equations (1)-(2)-(3)-(4) in spherical coordinates (r, θ, ϕ) and take the following fields, distributed on the tangent planes of spherical wave-fronts:

$$\mathbf{E} = \frac{1}{r}\Big(0,\ E_2(\theta,\phi),\ E_3(\theta,\phi)\Big)g(r-ct) \tag{8}$$

$$\mathbf{B} = \frac{1}{r}\Big(0,\ B_2(\theta,\phi),\ B_3(\theta,\phi)\Big)g(r-ct)$$

With this choice, the energy density $|\mathbf{E}|^2 + |c\mathbf{B}|^2$ remains constant when integrated over any spherical surface.

7. Using the curl of a vector in spherical coordinates, show that:

$$E_3 = -cB_2 \qquad E_2 = cB_3 \qquad (B_3 \sin\theta)_\theta = B_{2,\phi} \qquad (E_3 \sin\theta)_\theta = E_{2,\phi} \tag{9}$$

Hence, the complex functions $E_2 - iE_3 \sin\theta$ and $B_2 - iB_3 \sin\theta$ are holomorphic on the Riemann sphere. Deduce that **E** and **B** are bounded if and only if they are zero (see [1]). Similarly, deduce all the other consequences of the previous exercises.

Moral: The only possible solutions of Maxwell's equations in vacuum, having bounded electromagnetic fields laying on tangent planes of spherical fronts are identically zero. This is true despite common belief. The same arguments can be applied to any closed, bounded, compact, oriented surface. Non trivial solutions (for example, the well-known Hertz solution) are instead possible by allowing the fields to have radial components E_1 and B_1. In this way, however, the Poynting vector $\mathbf{E} \times \mathbf{B}$ is not lined up with the direction of movement. Indeed, the Hertz solution does not comply with the rules of geometrical optics.

Part III

Replace Maxwell's equations with the new system:

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \,\mathrm{curl}\mathbf{B} - (\mathrm{div}\mathbf{E})\mathbf{V} \tag{10}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\,\mathrm{curl}\mathbf{E} \tag{11}$$

$$\mathrm{div}\mathbf{B} = 0 \tag{12}$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{0} \tag{13}$$

where $\mathbf{V}$ is a velocity field. The system (10)-(13) satisfies all the basic requests imposed by physics, such as the existence of a Lagrangian. They can be written in covariant form and are invariant under Lorentz's transformations. More details can be found in [2] and [3].

8. Prove that, under the conditions:

$$E_1 = cB_2 \qquad E_2 = -cB_1 \qquad B_{2,x} = B_{1,y} \tag{14}$$

the fields proposed in (5) are always solutions, when $\mathbf{V} = (0, 0, c)$. The function g can be arbitrary. The impositions in (14) are clearly less restrictive than those in (6).

9. Prove that, under the conditions:

$$E_3 = -cB_2 \qquad E_2 = cB_3 \qquad (B_3 \sin\theta)_\theta = B_{2,\phi} \tag{15}$$

the fields proposed in (8) are always solutions, when $\mathbf{V} = (c, 0, 0)$. The function g can be arbitrary. The impositions in (15) are clearly less restrictive than those in (9).

10. From the settings (14) and (15) deduce that $\mathbf{E}$ is orthogonal to $\mathbf{B}$, and both are orthogonal to $\mathbf{V}$ (this is also consequence of (13)). Thus, the electromagnetic fields are transverse, i.e. they locally belong to the tangent plane of the propagation wave-fronts. In addition, we have $\|\mathbf{E}\| = \|c\mathbf{B}\|$.

11. Note that in the above examples we have $\|\mathbf{V}\| = c$. By writing $\mathbf{V}$ as a gradient of a potential Φ, we get $\|\nabla\Phi\| = c$. The last relation is the eikonal equation, that rules the dynamics of optical wave-fronts (see [4]).

Moral: With the new model equations we have an extended range of solutions, not available in the Maxwellian case. In particular, this includes wave-packets of almost any form, both from the viewpoint of the shape of the wave-fronts and the information written on them. These wave-packets travel unperturbed at speed c, along the directions determined by the vector field $\mathbf{V}$. They perfectly follow the rules of geometrical optics, a property that in the Maxwell's case is only ensured to plane waves of infinite extent and energy. In the extended model, there are not conditions on the divergence of $\mathbf{E}$, which can be different from zero even in vacuum (in absence of physical charges). This is the price to pay if we want to model electromagnetic emissions that behave like both waves and particles. Requiring $\text{div}\mathbf{E} = 0$ brings back to Maxwell's equations, where, as shown before, the solution space is practically empty.

[1] J. B. Conway, *Functions of One Complex Variable*, Second Edition, Springer, 1978.

[2] D. Funaro, *Electromagnetism and the Structure of Matter*, World-Scientific, Singapore, 2008.

[3] D. Funaro, *From Photons to Atoms -The Electromagnetic Nature of Matter*, World-Scientific, Singapore, 2019.

[4] M. Born & E. Wolf, *Principles of Optics*, 4th edition, Pergamon Press, 1980.